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Some extremal problems on subordinate functions

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ABSTRACT

Let A^p denote the Bergman space of functions f analytic in the unit disk D with

$$\|f\|_{A^p} = \left\{ \frac{1}{\pi} \int_D |f(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}} < +\infty,$$

where $p > 0$, and $\mathrm{d}A$ is the Lebesgue area measure. In this paper we investigate some extremal problems on subordinate A^p functions.

1. Introduction

Let $\mathcal{A}$ denote the locally convex linear topological space of functions analytic in the unit disk $D = \{z : |z| < 1\}$, with the topology of uniform convergence on compact subsets of the unit disk D. Let B_0 denote the set of functions $\phi \in \mathcal{A}$ such that $|\phi(z)| \le |z|$ for all $z \in D$. Let $f, F \in \mathcal{A}$. Then f is said to be subordinate to F, denoted by $f \prec F$, if and only if there exists a function $\phi \in B_0$ such that $f = F \circ \phi$. For any $F \in \mathcal{A}$, $s(F)$ denotes the family of all functions subordinate to F.

Let X be a linear topological space and let U be a subset of X. A point $u \in X$ is called an extreme point of U if $u \in U$ and if $u = tx + (1 - t)y$, where $0 < t < 1$, $x, y \in U$, implies that $x = y$. We use the notation EU to denote the set of extreme points of U.

For $0 < p < +\infty$, the Hardy space H^p is the set of functions $f \in \mathcal{A}$ with

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < +\infty,$$

where

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \mathrm{d}\theta \right\}^{\frac{1}{p}}.$$

For $0 < p < +\infty$, the Bergman space A^p is the set of functions $f \in \mathcal{A}$ with

$$\|f\|_{A^p} = \left\{ \frac{1}{\pi} \int_D |f(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}} < +\infty,$$

where $\mathrm{d}A$ denotes the Lebesgue area measure.

Ryff [1] proved that if $0 < p < \infty$, $F \in H^p$, $f \prec F$, then the necessary and sufficient condition for $\|f\|_{H^p} = \|F\|_{H^p}$ is $f = F \circ \phi$ where ϕ is inner and $\phi(0) = 0$. In [2] it is proved that $Hs(F) \subset H^p$ if $F \in H^p$ and $1 \le p < \infty$. Also in [2] it is proved that if $p > 1$, $f \in H^p$ and $F \ne 0$, $f \in Hs(F)$ and $\|f\|_{H^p} = \|F\|_{H^p}$, then $f \in EHs(F)$.

In this paper we consider similar problems on the Bergman space A^p.

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2. Main results

Theorem 1. *Suppose $f, g \in \mathcal{A}$, $f \prec g$, then*

$$\frac{1}{\pi}\int_{|z|\le R}|f(z)|^p \mathrm{d}A(z) \le \frac{1}{\pi}\int_{|z|\le R}|g(z)|^p \mathrm{d}A(z) \tag{1}$$

and equality holds if and only if g is a constant or $f(z) = g(xz)$, $|x| = 1$, where $0 < p < \infty$ and $0 < R < 1$.

Proof. Since $f \prec g$, it follows that

$$\int_0^{2\pi}|f(re^{i\theta})|^p \mathrm{d}\theta \le \int_0^{2\pi}|g(re^{i\theta})|^p \mathrm{d}\theta \tag{2}$$

where $0 \le r < 1$, $0 < p < \infty$, and the equality in (2) holds for some r if and only if there exists a number x with $|x| = 1$ such that $f(z) = g(xz)$ [3]. If $0 \le R < 1$, then

$$\begin{aligned}\frac{1}{\pi}\int_{|z|\le R}|f(z)|^p \mathrm{d}A(z) &= \frac{1}{\pi}\int_0^R\left[\int_0^{2\pi}|f(re^{i\theta})|^p \mathrm{d}\theta\right] r\mathrm{d}r\\ &\le \frac{1}{\pi}\int_0^R\left[\int_0^{2\pi}|g(re^{i\theta})|^p \mathrm{d}\theta\right] r\mathrm{d}r = \frac{1}{\pi}\int_{|z|\le R}|g(z)|^p \mathrm{d}A(z).\end{aligned}$$

Noticing that $\int_0^{2\pi}|f(re^{i\theta})|^p \mathrm{d}\theta = 2\pi M_p^p(r, f)$ and $\int_0^{2\pi}|g(re^{i\theta})|^p \mathrm{d}\theta = 2\pi M_p^p(r, g)$ are continuous with variable $r \in [0, 1)$ and the inequality (2), we conclude that if

$$\frac{1}{\pi}\int_{|z|\le R}|f(z)|^p \mathrm{d}A(z) = \frac{1}{\pi}\int_{|z|\le R}|g(z)|^p \mathrm{d}A(z)$$

then

$$\int_0^{2\pi}|f(re^{i\theta})|^p \mathrm{d}\theta = \int_0^{2\pi}|g(re^{i\theta})|^p \mathrm{d}\theta, \quad 0 < r \le R.$$

Thus $f(z) = g(xz)$ for some x with $|x| = 1$.

Conversely, if $f(z) = g(xz)$ with $|x| = 1$, then it is clear that

$$\frac{1}{\pi}\int_{|z|\le R}|f(z)|^p \mathrm{d}A(z) = \frac{1}{\pi}\int_{|z|\le R}|g(z)|^p \mathrm{d}A(z). \quad \square$$

Corollary 1. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}$, $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{A}$ and $f \prec g$, then*

$$\sum_{n=0}^{\infty}\frac{1}{n+1}|a_n|^2 r^{2n+2} \le \sum_{n=0}^{\infty}\frac{1}{n+1}|b_n|^2 r^{2n+2}$$

for any $r \in [0, 1)$.

Proof. Let $p = 2$. Then the conclusion follows from Theorem 1. $\square$

Corollary 2. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}$, $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{A}$ and $f \prec g$, then*

$$\sum_{n=0}^{N}\frac{1}{n+1}|a_n|^2 r^{2n+2} \le \sum_{n=0}^{N}\frac{1}{n+1}|b_n|^2 r^{2n+2},$$

and

$$\sum_{n=0}^{N}\frac{1}{n+1}|a_n|^2 \le \sum_{n=0}^{N}\frac{1}{n+1}|b_n|^2,$$

where $r \in [0, 1)$ and N is an arbitrary positive integer.

Proof. Since $f \prec g$, there exists a function $\phi \in B_0$ such that $f(z) = g(\phi(z))$, and so, there exist numbers c_n such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n[\phi(z)]^n = \sum_{n=0}^{N} b_n[\phi(z)]^n + \sum_{n=N+1}^{\infty} c_n z^n.$$

Thus there exist numbers d_n such that

$$\sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} d_n z^n = \sum_{n=0}^{N} b_n[\phi(z)]^n.$$

It follows from Corollary 1 that

$$\sum_{n=0}^{N} \frac{|a_n|^2}{n+1} r^{2n+2} + \sum_{n=N+1}^{\infty} \frac{|d_n|^2}{n+1} r^{2n+2} \leq \sum_{n=0}^{N} \frac{|b_n|^2}{n+1} r^{2n+2}.$$

Therefore

$$\sum_{n=0}^{N} \frac{|a_n|^2}{n+1} r^{2n+2} \leq \sum_{n=0}^{N} \frac{|b_n|^2}{n+1} r^{2n+2}. \tag{3}$$

Let $r \to 1$; from (3) we get

$$\sum_{n=0}^{N} \frac{|a_n|^2}{n+1} \leq \sum_{n=0}^{N} \frac{|b_n|^2}{n+1}. \quad \square$$

It must be pointed out that Corollaries 1 and 2 are due to G. Goluzin (in more general version even) [3].

Corollary 3. *If $f \prec g$ and $g \in A^p$ then $f \in A^p$, $\|f\|_{A^p} \leq \|g\|_{A^p}$ and the equality holds if and only if $f(z) = g(xz)$ with $|x| = 1$.*

Proof. It follows from (1) that

$$\sup_{0\leq R<1} \left\{ \frac{1}{\pi} \int_{|z|\leq R} |f(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}} \leq \sup_{0\leq R<1} \left\{ \frac{1}{\pi} \int_{|z|\leq R} |g(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}}.$$

That is $\|f\|_{A^p} \leq \|g\|_{A^p}$. Thus $g \in A^p$ implies that $f \in A^p$.

If $\|f\|_{A^p} = \|g\|_{A^p}$, then

$$\frac{1}{\pi} \int_0^1 \left[\int_0^{2\pi} |f(re^{i\theta})|^p \mathrm{d}\theta \right] r\mathrm{d}r = \frac{1}{\pi} \int_0^1 \left[\int_0^{2\pi} |g(re^{i\theta})|^p \mathrm{d}\theta \right] r\mathrm{d}r. \tag{4}$$

Since

$$2\pi M_p^p(r,f) = \int_0^{2\pi} |f(re^{i\theta})|^p \mathrm{d}\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^p \mathrm{d}\theta = 2\pi M_p^p(r,g)$$

and $M_p^p(r,f)$, $M_p^p(r,g)$ are continuous functions with variable $r \in [0, 1)$, it follows from (4) that

$$\int_0^{2\pi} |f(re^{i\theta})|^p \mathrm{d}\theta = \int_0^{2\pi} |g(re^{i\theta})|^p \mathrm{d}\theta$$

for $r \in [0, 1)$. Thus $f(z) = g(xz)$, where $|x| = 1$ [3].

Conversely, if $f(z) = g(xz)$ with $|x| = 1$ then it is obvious that $\|f\|_{A^p} = \|g\|_{A^p}$. $\square$

Theorem 2. *If $F \in A^p$, $1 \leq p < \infty$, then $Hs(F) \subset A^p$.*

Proof. Suppose that $f \in Hs(F)$. Then f can be uniformly approximated on each compact subset of D by functions of the form $\sum_{k=1}^{n} \lambda_k f_k$ where $f_k \in s(F)$, $0 \leq \lambda_k \leq 1$ and $\sum_{k=1}^{n} \lambda_k = 1$. Since $p \geq 1$ and

$$\frac{1}{\pi} \int_{|z|<1} |f_k(z)|^p \mathrm{d}A(z) \leq \frac{1}{\pi} \int_{|z|<1} |F(z)|^p \mathrm{d}A(z),$$

we may apply the Minkowski inequality to obtain the inequality

$$\begin{aligned} \left\{ \frac{1}{\pi} \int_{|z|<1} |\sum_{k=1}^{n} \lambda_k f_k(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}} &\leq \sum_{k=1}^{n} \lambda_k \left\{ \frac{1}{\pi} \int_{|z|<1} |f_k(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}} \\ &\leq \left\{ \frac{1}{\pi} \int_{|z|<1} |F(z)|^p \mathrm{d}A(z) \right\}^{\frac{1}{p}}. \end{aligned}$$

Therefore $f \in A^p$ and $\|f\|_{A^p} \leq \|F\|_{A^p}$. $\square$

Let $B(A^p) = \{f : f \in A^p, \|f\|_{A^p} \leq 1\}$. If $p > 1$ then A^p is a uniformly convex space. It is known that $EB(A^p) = \{f : f \in A^p, \|f\|_{A^p} = 1\}$ when $p > 1$ [4].

Theorem 3. *Suppose that $p > 1$, $F \in A^p$, $F \neq 0$. If $f \in Hs(F)$ and $\|f\|_{A^p} = \|F\|_{A^p}$, then $f \in EHs(F)$.*

Proof. Suppose that $f \in Hs(F)$ and $\|f\|_{A^p} = \|F\|_{A^p}$, and there exist functions g and h in $Hs(F)$ and a number t $(0 < t < 1)$ such that $f = tg + (1-t)h$. Since $\|g\|_{A^p} \leq \|F\|_{A^p}$ and $\|h\|_{A^p} \leq \|F\|_{A^p}$, it follows that $\|f\|_{A^p} = \|g\|_{A^p} = \|h\|_{A^p} = \|F\|_{A^p}$. We note that $\|F\|_{A^p} \neq 0$ and so $f_1 = \frac{f}{\|F\|_{A^p}}$, $g_1 = \frac{g}{\|F\|_{A^p}}$ and $h_1 = \frac{h}{\|F\|_{A^p}} \in \partial B(A^p) = \{\varphi \in A^p, \|\varphi\|_{A^p} = 1\}$. Since $f_1 = tg_1 + (1-t)h_1$, $0 < t < 1$ and $f_1 \in EB(A^p)$, it follows that $f_1 = g_1 = h_1$, and so $f = g = h$. This implies that $f \in EHs(F)$. $\square$

Theorem 4. *Suppose that $p > 1$, $F \in A^p$. If $f \in Hs(F)$ and $\|f\|_{A^p} = \|F\|_{A^p}$ then $f(z) = F(xz)$ where $|x| = 1$.*

Proof. Since $\|f\|_{A^p} = \|F\|_{A^p}$, it follows from Theorem 3 that $f \in EHs(F)$. Since $Hs(F)$ is a compact set, $EHs(F) \subset s(F)$ [2]. So $f \in s(F)$. It follows from Corollary 3 that $f(z) = F(xz)$ with $|x| = 1$. $\square$

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